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A Note on Rational L_p Approximation

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1. INTRODUCTION

In a recent paper, J. H. Freilich has given a characterization of a locally best rational L_p approximation using convexity techniques which have usually been associated with approximation in the uniform norm. The purpose of this short note is to point out that Freilich's characterization theorem holds true in the setting of osculatory interpolation.

Let $\{y_1, ..., y_k\}$ be a fixed set of k points in (0, 1) and $\{m_1, ..., m_k\}$ a fixed set of positive integers, and set $s = \max\{m_i - 1 : 1 \leqslant i \leqslant k\}$. Let C^s denote the set consisting of those real functions on (0, 1) whose sth derivative is continuous there, and let P and Q denote finite-dimensional subspaces of C^s. Let f and r_0 belong to C^s. Define sets R, $R(f)$ and $K(r_0)$ as follows:

$$R = \{r = \tilde{p}/\tilde{q} : \tilde{p} \in P,\ \tilde{q} \in Q,\ \tilde{q} > 0 \text{ on } (0, 1)\},$$

$$R(f) = \{r \in R : r^{(j)}(y_i) = f^{(j)}(y_i),\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant k\},$$

$$K(r_0) = \{h \in P + r_0 Q : h^{(j)}(y_i) = 0,\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant k\}.$$

Now, suppose $n = \dim P$, $t = \dim Q$ and $\{g_1, ..., g_n\}$ and $\{h_1, ..., h_t\}$ are bases for P and Q, respectively. Let D denote a bounded set in E^{n+t} which contains the origin and let $\| v \|$ denote the Euclidean norm for $v \in E^{n+t}$.

We say that r_0 is a locally best rational L_p $(1 \leqslant p < \infty)$ approximation to f in C^s from $R(f)$ with respect to D (LBA) if there exists an $\epsilon_0 > 0$ so that for all $(a, b) \in D$ such that $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_t)$, $\|(a, b)\| \leqslant \epsilon_0$ and $\tilde{p}/\tilde{q} \in R(f)$ where $\tilde{p} = \sum a_i g_i$, $\tilde{q} = \sum b_i h_i$, we have $\|f - r_0\| \leqslant \|f - r_\lambda\|$ for all λ such that $0 < |\lambda| \leqslant \epsilon_0$ and $q_0 + \lambda\tilde{q} > 0$ on (0, 1), where $r_\lambda = (p_0 + \lambda\tilde{p})/(q_0 + \lambda\tilde{q})$.

We observe that $r_\lambda - r_0 = \lambda h/(q_0 + \lambda\tilde{q})$, where $h = \tilde{p} - r_0\tilde{q}$. Since $h \in K(r_0)$ if $r_0 \in R(f)$ and $\tilde{p}/\tilde{q} \in R(f)$ (see Perrie [2]), we see that $r_\lambda \in R(f)$ for all allowable values of λ.

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2. Characterization

Let S denote the unit sphere in $L_p^*(0, 1)$ and let ext(S) denote the set of extreme points of S. Let $E_0(S) = \{L \in \text{ext } S: L(f - r_0) = \|f - r_0\|\}$.

THEOREM. *Let* $\{\phi_1, ..., \phi_N\}$ *be a basis for* $K(r_0)$, $r_0 = p_0/q_0$, *and let* $v(L) = (L(\phi_1/q_0), ..., L(\phi_N/q_0))$.

(a) *If* r_0 *is an LBA, then* $0 \in H\{v(L): L \in E_0(S)\}$.

(b) *If* $0 \in H^0\{v(L): L \in E_0(S)\}$, *then* r_0 *is an LBA.*

(Here, H denotes the convex hull and H^0 its interior.)

Proof. Since the proof follows directly from the work of Freilich we only establish part (b). Suppose r_0 is not an LBA. Then for each $\epsilon > 0$ there exists a $\lambda = \lambda(\epsilon)$, $0 < |\lambda| \leqslant \epsilon$, and coefficient vectors $c = c(\epsilon)$ and $d = d(\epsilon)$ such that $(c, d) \in D$, $\|(c, d)\| \leqslant \epsilon$, $q_0 + \lambda\tilde{q} > 0$ on $(0, 1)$ and $\|f - r_\lambda\| < \|f - r_0\|$, where $\tilde{p} = \sum c_i g_i$, $\tilde{q} = \sum d_i h_i$, $\tilde{p}/\tilde{q} \in R(f)$ and $r_\lambda = (p_0 + \lambda\tilde{p})/(q_0 + \lambda\tilde{q})$. Let $L \in E_0(S)$. Then $L(r_\lambda - r_0) = L(f - r_0) - L(f - r_\lambda) > 0$. Let $h = \tilde{p} - r_0\tilde{q}$. Then $h \in K(r_0)$ and $r_\lambda - r_0 = \lambda h/(q_0 + \lambda\tilde{q})$. So,

$$L(r_\lambda - r_0) = \lambda L(h/(q_0 + \lambda\tilde{q})) = \sum \lambda a_i L(\phi_i/(q_0 + \lambda\tilde{q}))$$

for some set of real numbers $\{a_1, ..., a_N\}$. Therefore $0 \notin H\{w(L): L \in E_0(S)\}$ where $w(L) = (L(\phi_1/(q_0 + \lambda\tilde{q})), ..., L(\phi_N/(q_0 + \lambda\tilde{q})))$. However, $0 \in H^0\{v(L): L \in E_0(S)\}$ implies that $0 \in H\{w(L): L \in E_0(S)\}$ if $v(L)$ and $w(L)$ are sufficiently close [1]. Therefore the result follows.

COROLLARY. (a) *A necessary condition for* $r_0 = p_0/q_0$ *to be an LBA is that* $\min\{L(h/q_0): L \in E_0(S)\} \leqslant 0$ *for all* $h \in K(r_0)$. (b) *A sufficient condition for* r_0 *to be an LBA is that* $\min\{L(r_\lambda - r_0): L \in E_0(S)\} \leqslant 0$ *for all* r_λ *satisfying the conditions of the definition of an LBA.*

We observe that the sufficiency is equivalent to the condition that min $\{L(h/q_0 + \lambda\tilde{q}): L \in E_0(S)\} \leqslant 0$ for all $h = \tilde{p} - r_0\tilde{q}$ in $K(r_0)$ for which $\tilde{p}$ and $\tilde{q}$ satisfy the conditions of the definition of an LBA.

References

1. J. H. Freilich, Generalized rational approximation in interpolating subspaces of $L_p(\mu)$, *J. Approximation Theory* **9** (1973), 380–394.
2. A. L. Perrie, Uniform rational approximation with osculatory interpolation, *J. Comput. System Sci.* **4** (1970), 509–522.